Homework 5: Solutions to exercises not appearing in Pressley

Math 120A

- (3.2.2) and (3.2.3) Parametrize the ellipse as $\gamma(t) = (p\cos t, q\sin t)$. Then the area of the ellipse is

$$\begin{aligned} A(\gamma) &= \frac{1}{2}\int_\gamma xdy - ydx \\ &= \frac{1}{2}\int_0^{2\pi} [pq\cos^2 t - (-pq\sin^2 t)]dt \\ &= \frac{1}{2}\int_0^{2\pi} pqdt \\ &= \pi pq \end{aligned}$$

Furthermore, the length $\ell(\gamma)$ of the ellipse is $\int_0^{2\pi} \sqrt{p^2\sin^2 t + q^2\cos^t} dt$. The isoperimetric inequality tells us that $A(\gamma) \leq \frac{\ell(\gamma)^2}{4\pi}$, or $2\sqrt{\pi A(\gamma)} \leq \ell(\gamma)$. Therefore we have

$$\begin{aligned} \int_0^{2\pi} \sqrt{p^2\sin^2 t + q^2\cos^t dt} &\geq 2\sqrt{\pi(\pi pq)} \\ &= 2\pi\sqrt{pq} \end{aligned}$$

Since the isoperimetric inequality is an equality if and only if γ is a circle, the last inequality is equality if and only if the ellipse is a circle, that is, if $p = q$.

- (3.3.4) Let $\gamma(x) = (x, f(x))$. Then the tangent vector $\dot{\gamma}(x) = (1, \dot{f})$, and the length of the tangent vector is $\sqrt{1 + (\dot{f})^2}$. Therefore the unit tangent vector of γ is

$$\mathbf{t} = \left(\frac{1}{1 + (\dot{f})^2}\right)(1, \dot{f}).$$

Rotating by $\frac{\pi}{2}$ shows that the signed unit normal is

$$\mathbf{n}_s = = \left(\frac{1}{1 + (\dot{f})^2}\right)(-\dot{f}, 1).$$

Recall that κ_s the signed curvature is the number that makes $\frac{d\mathbf{t}}{dv} = \kappa_s \mathbf{n}_s$, where v is the arc length. Ergo we compute:

$$\begin{aligned}
\frac{d\mathbf{t}}{dv} &= \frac{dx}{dv}\frac{d}{dx}\left(\frac{1}{1+(\dot{f})^2}, \frac{\dot{f}}{1+(\dot{f})^2}\right) \\
&= \frac{1}{\sqrt{1+(\dot{f})^2}}\left(\frac{\frac{-\dot{f}\ddot{f}}{\sqrt{1+(\dot{f})^2}}}{1+(\dot{f})^2}, \frac{\ddot{f}\sqrt{(1+(\dot{f})^2} - \dot{f}\frac{-\dot{f}\ddot{f}}{\sqrt{1+(\dot{f})^2}}}{1+(\dot{f})^2}\right) \\
&= \frac{1}{\sqrt{1+(\dot{f})^2}}\left(\frac{-\dot{f}\ddot{f}}{(1+(\dot{f})^2)^{\frac{3}{2}}}, \frac{\ddot{f}(1+(\dot{f})^2) - (\dot{f}^2)\ddot{f}}{(1+(\dot{f})^2)^{\frac{3}{2}}}\right) \\
&= \frac{\ddot{f}}{(1+(\dot{f})^2)^{\frac{3}{2}}}\left(\frac{1}{\sqrt{1+(\dot{f})^2}}(-\dot{f}, 1)\right) \qquad = \frac{\ddot{f}}{(1+(\dot{f})^2)^{\frac{3}{2}}}\mathbf{n}_s
\end{aligned}$$

We see that $\kappa_s = \frac{\ddot{f}}{(1+(\dot{f})^2)^{\frac{3}{2}}}$. This implies that the numerator of $\dot{\kappa}_s$ is

$$\dddot{f}(1+(\dot{f}^2))^{\frac{3}{2}} - \ddot{f}(2\dot{f}\ddot{f})\left(\frac{3}{2}\right)(1+(\dot{f})^2)^{\frac{1}{2}}$$

which we rearrange to

$$(1+(\dot{f})^2)^1 2\left(\dddot{f}(1+(\dot{f})^2)) - 3(\ddot{f})^2\dot{f}\right)$$

Now $(1+(\dot{f})^2)^1 2$ is never zero, so if $\dot{\kappa}_s = 0$, we have $\dddot{f}(1+(\dot{f})^2)) = 3(\ddot{f})^2\dot{f}$. This was the desired equality.

- (4.1.7) Recall that in class we gave a surface patch for the unit sphere defined on $U = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi)$ with map $\sigma(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$. We can adapt this to be a surface patch for the ellipsoid by replacing σ with σ' such that $\sigma'(\theta, \phi) = (p\cos\theta\cos\phi, q\cos\theta\sin\phi, r\sin\phi)$. The inverse of this map is $\sigma^{-1}(x, y, z) = (\arcsin\frac{z}{r}, \arctan\left(\frac{py}{qx}\right))$ on $\{x > 0, y \neq 0\}$, $\sigma^{-1}(x, y, z) = (\pi + \arctan\left(\frac{px}{qy}\right), \arcsin\left(\frac{z}{r}\right))$ on $\{x < 0\}$, and $\sigma^{-1}(x, y, z) = (\cot^{-1}\left(\frac{qy}{px}\right), \arcsin\left(\frac{z}{r}\right))$ on $\{x > 0\}$, and $\sigma^{-1}(x, y, z) = (\pi + \cot^{-1}\left(\frac{px}{qy}\right), \arcsin\left(\frac{z}{r}\right))$ on $\{x < 0\}$. These pieces all fit together to form a continuous inverse function. The question only asked for one surface patch, but doing the same change to the other surface patch for the sphere produces an atlas for the sphere. (Alternately, one can adapt the charts given in Exercise 4.1.2.)

- (4.1.8) We have $\sigma(u, v) = (\sin u, \sin v, \sin(u+v))$, which is an injection on $-\frac{\pi}{2} < u, v < \frac{\pi}{2}$. Moreover, $\sigma^{-1})(x, y, z) = (\arcsin x, \arcsin y)$ is also continuous, so σ is a continuous bijection (onto its image) with continuous inverse. It remains to be shown that the image of σ is on the Cartesian surface $(x^2 + y^2 + z^2)^2 = 4x^2z^2(1 - y^2)$, i.e., that points $\sin u, \sin v, \sin(u+v)$ satisfy this equation. Notice that the lefthand side above, upon substitution, becomes $4\sin^2 u \sin^2(u+v)(1-\sin^2 v) = 4\sin^2 u \sin^2(u+y)\cos^2 v = (2\sin u \sin(u+v)\cos v)^2$. Therefore it suffices to check that $\sin^2 u - \sin^2 v + \sin^2(u+v) = 2\sin u \sin(u+v)\cos v$. Let's do this:

$$\begin{aligned}
\sin^2 u - \sin^2 v + \sin^2(u+v) &= \sin^2 u - \sin^2 v + (\sin u \cos v + \cos u \sin v)^2 \\
&= \sin^2 u - \sin^2 v + \sin^2 u \cos^2 v + 2\sin u \cos u \sin v \cos v + \sin^2 v \cos^2 u \\
&= \sin^2 u(1+\cos^2 v) + \sin^2 v(\cos^2 u - 1) + 2\sin u \cos u \sin v \cos v \\
&= \sin^2 u(1+\cos^2 v) + \sin^2 v \sin^2 u + 2\sin u \cos u \sin v \cos v \\
&= \sin^2 u(1+\cos^2 v - \sin^2 v) + 2\sin u \cos u \sin v \cos v \\
&= 2\sin^2 u \cos^2 v + 2\sin u \cos u \sin v \cos v \\
&= 2\sin u \cos v(\sin u \cos v + \cos u \sin v) \\
&= 2\sin u \cos v \sin(u+v)
\end{aligned}$$

- Question 3: Stereographic projection. Let σ_1 be the projection map from the sphere to the plane by drawing lines through the north pole. If (x,y,z) lies on the unit sphere, the line through $(0,0,1)$ and (x,y,z) is given by $(0,0,1)+t(x,y,(z-1))$. This line intersects the xy-plane at $0 = 1+t(z-1)$, or $t = \frac{1}{1-z}$. Therefore the projection map is

$$\begin{aligned}
\sigma_1 : S - \{(0,0,1)\} &\to \mathbb{R}^2 \\
(x,y,z) &\mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)
\end{aligned}$$

This is clearly smooth. Now let us determine σ^{-1}. The line through the north pole and the point $(u,v,0)$ on the (x,y) plane is $(0,0,1)+t(u,v,-1)$. We want to find the point on this line that lies on the unit sphere $x^2+y^2+z^2=1$. Therefore we take an arbitrary point $(tu, tv, 1-t)$ on this line and solve:

$$\begin{aligned}
(tu)^2 + (tv)^2 + (1-t)^2 &= 1 \\
t^2u^2 + t^2v^2 + 1 - 2t + t^2 &= 1 \\
t(tu^2 + tv^2 + t - 2) &= 0 \\
t &= \frac{2}{u^2+v^2+1}
\end{aligned}$$

Thus the map σ^{-1}(which is the actual surface patch) is

$$\sigma_1^{-1} : \mathbb{R}^2 \to S - \{(0,0,1)\}$$
$$(u,v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

This is the map whose regularity we need to check. The partial derivatives of σ^{-1} with respect to u and v are

$$\frac{d(\sigma_1^{-1})}{du} = \frac{2}{(u^2+v^2+1)^2}(v^2-u^2+1, -2uv, 2u)$$
$$\frac{d(\sigma_1^{-1})}{dv}\frac{2}{(u^2+v^2+1)^2}(-2uv, u^2-v^2+1, 2v)$$

To show these vectors are linearly independent, we need to check that the cross product of $(v^2 - u^2 + 1, -2uv, 2u)$ and $(-2uv, u^2 - v^2 + 1, 2v)$ is nonzero. The first entry in this vector is $-4uv^2 - 4u(u^2 - v^2 + 1) = -4u(u^2 + 1) \neq 0$ unless $u = 0$. Symmetrically, the second entry is only zero if $v = 0$. However, the third entry is equal to $1 - (v^2 + u^2)^2$, and is only zero if $u^2 + v^2 = 1$. These three things cannot happen simultaneously. So regularity holds.

Finally, we consider the transition maps. If we define a second projection map $\sigma_2 : S - \{(0, 0, -1)\} \to \mathbb{R}^2$ given by $(x, y, z) \mapsto (\frac{x}{1+z}, \frac{y}{1+z}\}$. Then the transition map $\sigma_2 \circ \sigma_1^{-1}$ is the map $\mathbb{R}^2 - \{(0, 0\} \to \mathbb{R}^2 - \{(0, 0)\}$ given by $(u, v) \to \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$. Notice that if we think about $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ instead, this is the map $z \mapsto \frac{1}{z}$.

- (4.1.3) Let S be the hyperboloid $\{(x, y, z) : x^2 + y^2 - z^2 = 1\}$. Then we claim the lines L_θ defined by

$$\begin{aligned}(x - z)\cos\theta &= (1 - y)\sin\theta \\ (x + z)\sin\theta &= (1 + y)\cos\theta\end{aligned}$$

for fixed θ on the hyperboloid. For multiplying the equations gives

$$(x^2 - z^2)\sin\theta\cos\theta = (1 - y^2)\sin\theta\cos\theta$$

Therefore either $x^2 + -z^2 = 1 - y^2$, which is the equation of the hyperboloid, or one of $\sin\theta$ or $\cos\theta$ is zero. If $\sin\theta$ is zero, then $x = z$ and $y = -1$, which is on the hyperboloid, and if $\cos\theta = 0$, then $y = 1$ and $x = -z$, which is similarly on the hyperboloid. Now we use this to give a parametrization of the hyperboloid in two surface patches. First, we want to describe the line L_θ more concretely. Notice that each L_θ passes through some point where $z = 0$; at that point, assuming θ is not a multiple of $\frac{\pi}{2}$, we have $x^2 + y^2 = 1$ and $(1 - y)\tan\theta = x = (1 + y)\cot\theta$. Note that the sign of $1 + y$ is the same as the sign of $1 - y$. Moreover we have $\tan^2\theta = \frac{1+y}{1-y}$, which implies that $\sin\theta = \pm\frac{\sqrt{|1+y|}}{2}$ and $\cos\theta = \pm\frac{\sqrt{|1-y|}}{2}$. We conclude that $\sin\theta\cos\theta = \pm\frac{\sqrt{1-y^2}}{2} = \frac{\sqrt{x^2}}{2} = \pm\frac{x}{2}$. We conclude that $x = \pm 2\sin\theta\cos\theta = \sin(2\theta)$, and therefore $y = \pm\cos(2\theta)$. The only sign choice that fits the equations of L_θ is $(x, y, 0) = (\sin(2\theta), -\cos(2\theta), 0)$. Now we know one point on L_θ.

The other easy-to-examine point on the line L_θ is $z = \cot\theta$. Then the second equation for L_θ becomes

$$\begin{aligned}(x + \cot\theta)(\sin\theta) &= (1 + y)\cos\theta \\ x\sin\theta + \cos\theta &= \cos\theta + y\cos\theta \\ x\sin\theta &= y\cos\theta\end{aligned}$$

and the first becomes, similarly

$$x\cos\theta + y\sin\theta = \csc\theta$$

The solution to this system of equations is $x = \csc\theta\cot\theta = \cot\theta$ and $y = \csc\theta\sin\theta = 1$). So the point $(\cot\theta, 1, \cot\theta)$ also lies on L_θ.

We are finally ready to give a single equation for L_θ. We know this line is parallel to the vector $(\cot\theta, 1, \cot\theta) - (\sin(2\theta, -\cos 2\theta, 0)]$. Multiplying by $\tan\theta$ (remember we are still not doing the case that θ is a multiple of $\frac{\pi}{2}$) we see our line is parallel to

$$\begin{aligned}(1 - \tan\theta\sin(2\theta), \tan\theta + \tan\theta\cos(2\theta), 1) &= \left(1 - \frac{\sin\theta}{\cos\theta}2\sin\theta\cos\theta, \frac{\sin\theta}{\cos\theta}(1 + (2\cos^2\theta - 1)), 1\right) \\ &= (1 - 2\sin^2\theta, 2\sin\theta\cos\theta, 1) \\ &= (\cos(2\theta), \sin(2\theta), 1)\end{aligned}$$

So L_θ is the line $(\sin(2\theta), -\cos(2\theta), 0) + t(\cos(2\theta), \sin(2\theta), 1)$, at least when θ is not a multiple of $\frac{\pi}{2}$. But when $\theta = 0$, we have the line $(t, -1, t)$, which also fits this description, and when $\theta = \frac{\pi}{2}$, we have the line $(t, 1, -t)$, which does as well. Hence we have completely described L_θ, and our hyperboloid has the parametrization $\sigma(\theta, t) = (\sin(2\theta), -\cos(2\theta), 0) + t(\cos(2\theta), \sin(2\theta), 1)$. Restricting σ to the domains $(0, \pi) \times \mathbb{R}$ and $(\frac{\pi}{2}, \frac{3\pi}{2}) \times \mathbb{R}$ gives two surface patches that cover the hyperboloid.

So far we have seen that the hyperboloid is a *ruled* surface; to wit, it is parametrized by a family of lines. Now we will see it is *doubly ruled*. There is another family of lines M_ϕ that twist in the opposite direction on the hyperboloid. We obtain them by changing the sign of ϕ.

$$\begin{aligned}(x - z)\cos\phi &= (1 + y)\sin\phi \\ (x + z)\sin\phi &= (1 - y)\cos\phi\end{aligned}$$

We want to show that M_ϕ and L_θ intersect in a single point unless $\theta + \phi = n\pi$ for θ and ϕ not multiples of $\frac{\pi}{2}$. If either is true we have $(1 - y)\sin\theta = -(1 + y)\sin\theta$, which has no solutions when $\sin\theta \neq 0$. If $\theta = \phi = 0$, then L_θ is the line $(t, -1, t)$ and M_ϕ is the line $(t, 1, t)$, which do not intersect; similarly if $\theta = \phi = \frac{\pi}{2}$.

Now consider the case that $\theta + \phi \neq \pi n$ and neither θ nor ϕ is a multiple of $\frac{\pi}{2}$. Notice that if (x, y, z) lies on both lines, we have $x - z = (1 + y)\tan\phi = (1 - y)\tan\theta$. If $\phi \neq -\theta$, and neither is a multiple of $\frac{\pi}{2}$, this implies

$$\begin{aligned}y &= \frac{\tan\theta - \tan\phi}{\tan\phi + \tan\theta} \\ &= \frac{\sin\theta\cos\phi - \sin\phi\cos\theta}{\sin\phi\cos\theta + \sin\theta\cos\phi} \\ &= \frac{\sin(\theta - \phi)}{\sin(\phi + \theta)}\end{aligned}$$

Moreover, we know that $2x = (1 + y)\tan\phi + (1 - y)\cot\phi$. Inputting the expression for y above and simplifying shows that $x = \frac{\cos(\theta - \phi)}{\sin(\theta + \phi)}$. But then $z = x - (1 - y)\tan\phi$, so inputting our formulas for x and y and simplifying shows that $z = \frac{\cos(\theta+\phi)}{\sin(\theta+\phi)}$. Ergo our intersection point is in general

$$\left(\frac{\cos(\theta-\phi)}{\sin(\theta+\phi)}, \frac{\sin(\theta-\phi)}{\sin(\phi+\theta)}, \frac{\cos(\theta+\phi)}{\sin(\theta+\phi)}\right)$$

We still need to deal with the case that one angle is 0 or $\frac{\pi}{2}$. If $\theta = 0$, then we have the line $(t, -1, t)$, which intersects M_ϕ where $2t \sin\phi = 2\cos\phi$, or $t = \cot(\phi)$ (as long as $\phi \neq 0$). This fits the general form above. If $\phi = \frac{\pi}{2}$ we get the line $(t, 1, -t)$, which interects M_ϕ where $t = -\tan\phi$, at $(-\tan\phi, 1, \tan\phi)$. This again, fits the general form. We see similar behaviour if ϕ is 0 or $\frac{\pi}{2}$.